

Selecting Probabilistic Approaches for Reliability-Based Design Optimization

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During the past decade, numerous endeavors have been made to develop effective reliability-based design optimization (RBDO) methods. Because the evaluation of probabilistic constraints defined in the RBDO formulation is the most difficult part to deal with, a number of different probabilistic design approaches have been proposed to evaluate probabilistic constraints in RBDO. In the first approach, statistical moments are approximated to evaluate the probabilistic constraint. Thus, this is referred to as the approximate moment approach (AMA). The second approach, called the reliability index approach (RIA), describes the probabilistic constraint as a reliability index. Last, the performance measure approach (PMA) was proposed by converting the probability measure to a performance measure. A guide for selecting an appropriate method in RBDO is provided by comparing probabilistic design approaches for RBDO from the perspective of various numerical considerations. It has been found in the literature that PMA is more efficient and stable than RIA in the RBDO process. It is found that PMA is accurate enough and stable at an allowable efficiency, whereas AMA has some difficulties in RBDO process such as a second-order design sensitivity required for design optimization, an inaccuracy to measure a probability of failure, and numerical instability due to its inaccuracy. Consequently, PMA has several major advantages over AMA, in terms of numerical accuracy, simplicity, and stability. Some numerical examples are shown to demonstrate several numerical observations on the three different RBDO approaches.

Nomenclature

d	= design parameter, $[d_1, d_2, \dots, d_n]^T$
d^L, d^U	= lower and upper bounds of design parameter d
$F_G(\cdot)$	= cumulative distribution function (CDF) of the performance function $G(X)$, $F_G(g) = P[G(X) < g]$
$f_x(\mathbf{x})$	= joint probability density function of the random parameter
G_p^{AMA}	= probabilistic constraint in approximate moment approach (AMA)
G_p^{PMA}	= probabilistic constraint in performance measure approach (PMA)
G_p^{RIA}	= probabilistic constraint in reliability index approach (RIA)
$G(X)$	= performance function; design is considered to fail if $G(X) < 0$
n	= normalized steepest descent direction of performance function
$P(\cdot)$	= probability function
U	= independent and standard normal random parameter, $[U_1, U_2, \dots, U_n]^T$
u	= realization of U , $[u_1, u_2, \dots, u_n]^T$
$u_{(A)MV}^*$	= most probable point (MPP) using (advanced) mean value method in PMA
u_{CMV}^*	= MPP using conjugate mean value method in PMA
u_{HMV}^*	= MPP using hybrid mean value method in PMA
$u_{\beta=\beta_i}^*$	= MPP in first-order inverse reliability analysis

X	= random parameter, $[X_1, X_2, \dots, X_n]^T$
x	= realization of X , $[x_1, x_2, \dots, x_n]^T$
β_{MCS}	= reliability obtained from Monte Carlo simulation (MCS)
β_s	= safety reliability index, $-\Phi^{-1}(P_f)$ and $-\Phi^{-1}[F_G(g)]$
β_t	= target reliability index
μ	= mean of random parameter X , $[\mu_1, \mu_2, \dots, \mu_n]^T$
ζ	= criteria for the type of performance function $G(X)$
$\Phi(\cdot)$	= standard normal CDF

I. Introduction

BECAUSE of the globally competitive marketplace, many industries have endeavored to improve the quality of their designs and processes for 6-sigma products. At the same time, significant advances in computing power give design engineers and design decision makers opportunity to explore many more alternative simulation-based designs than they could do with hardware alone. Therefore, many attempts were recently made to integrate a simulation-based design optimization into a probabilistic approach, developing several different probabilistic design approaches for reliability-based design optimization (RBDO). Consequently, the RBDO process provides not only cost-effective manufacturing processes but also their target confidence.

The probabilistic constraint is the key constraint in RBDO, the evaluation of which creates several numerical challenges associated with numerical efficiency, accuracy, stability, etc.¹⁻³ Accordingly, several different probabilistic design approaches have been proposed for RBDO: approximate moment approach (AMA),^{4,5} reliability index approach (RIA),⁶ performance measure approach (PMA)¹⁻³ (or fixed norm approach⁷), and sampling-based approach.^{8,9} The first approach was introduced by approximately matching statistical moments, referred to as AMA, which originated from the robust design optimization concept. In the second approach, RIA, the probabilistic constraint is viewed as a reliability index, which is originated from the reliability analysis concept. Third, the PMA, which originated from the reliability-based design concept, was proposed by converting the probability measure to a performance measure. Last, the sampling-based approach employs some sampling techniques such as Monte Carlo simulation, important sampling, and adaptive sampling techniques for RBDO. However, it is practically impossible to use the sampling-based approach because it is computationally too expensive.

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This paper presents observations on the first three different probabilistic design approaches for RBDO in terms of numerical accuracy, stability, and efficiency. It is found that AMA has several drawbacks compared to PMA and RIA. First, approximation of the standard deviation of the performance function is independent of the distribution type of the random input parameter. Second, the reliability assessment by the first two moments of performance produces large numerical error because AMA ignores the probabilistic distribution type of the performance. Third, the fact that the probabilistic constraint in AMA is approximated at the mean value point leads to larger error as the target reliability becomes higher. Fourth, AMA could be numerically very expensive because it requires the second-order sensitivity of the performance function to evaluate the sensitivity of the probabilistic constraint,^{4,5} whereas PMA and RIA require optimization problems to be solved for reliability analysis. Thus, it is difficult to describe accurately a required reliability target such as 6-sigma using the AMA method.

There are several numerical advantages of PMA as compared to RIA. First, the convergence of the reliability analysis using PMA is inherently more robust and efficient than that using RIA because it is easier to minimize a complicated function subject to a simple constraint function using PMA than to minimize a simple function subject to a complicated constraint function using RIA.¹⁻³ Second, the nonlinearity of the PMA reliability constraint is less dependent on probabilistic model types, such as normal, lognormal, Weibull, Gumbel, and uniform distributions, than the RIA reliability constraint.³ Thus, RIA tends to diverge for distributions other than the normal distribution, whereas PMA converges well for all types of distributions.

However, a major challenge still remains regarding the RBDO methodology in obtaining reliability-based optimum designs for multidisciplinary large-scale models using a limited number of design analyses. Moreover, for certain application areas, lack of design sensitivity information creates considerable difficulty in using RBDO methods. To overcome this difficulty, the RBDO methods are often integrated with the response surface method (RSM), which is composed of function representation techniques and design of experiment (DOE).⁹ The RSM is highly attractive for use in RBDO because limited design samples can produce response surface approximations for both reliability analysis and design optimization. When RSM is used for RBDO, PMA has been found to be substantially preferable to RIA, due to both the well-defined DOE building block size associated with the reliability target and the consistency between the design window and the reliability analysis window.¹⁰

Some numerical examples are shown to demonstrate some numerical observations on the three different RBDO approaches.

II. RBDO

In system parameter design, the RBDO model^{1-3,7} can generally be defined as

$$\begin{aligned} & \text{minimize} && \text{cost}(\mathbf{d}) \\ & \text{subject to} && P[G_i(\mathbf{X}) \geq 0] - \Phi(-\beta_i) \leq 0, \quad i = 1, \dots, np \\ & && \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \quad \mathbf{d} \in \mathbb{R}^{ndv}, \quad \mathbf{X} \in \mathbb{R}^{nr} \end{aligned} \quad (1)$$

where $\mathbf{d} = \boldsymbol{\mu}(\mathbf{X})$ is the design vector, \mathbf{X} is the random vector, and the probabilistic constraint is described by the performance function $G_i(\mathbf{X})$, its probability distributions, and its prescribed reliability target β_i .

The statistical description of the failure of the performance function $G_i(\mathbf{X})$ is characterized by the cumulative distribution function $F_{G_i}(0)$ as

$$P[G_i(\mathbf{X}) \geq 0] = 1 - F_{G_i}(0) \leq \Phi(-\beta_i) \quad (2)$$

where the reliability of failure is defined as

$$F_{G_i}(0) = \int_{G_i(\mathbf{X}) \leq 0} \dots \int f_{\mathbf{X}}(\mathbf{x}) \, d\mathbf{x} \quad (3)$$

In Eq. (3), $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function of all random parameters. Evaluation of the probabilistic constraint in Eq. (2) requires a reliability analysis where the multiple integration of Eq. (3) is involved. Some approximate probability integration methods have been developed to provide efficient solutions, such as the first-order reliability method (FORM) and the asymptotic second-order reliability method with a rotationally invariant measure as the reliability.^{11,12} FORM often provides adequate accuracy and is widely used for RBDO applications.^{1-3,7} In FORM, reliability analysis requires a transformation \mathbf{T} (Ref. 13) from the original random parameter \mathbf{X} to the independent and standard normal random parameter \mathbf{U} . That is, the performance function $G(\mathbf{X})$ in \mathbf{X} space is mapped onto $G[\mathbf{T}(\mathbf{X})] \equiv G(\mathbf{U})$ in \mathbf{U} space.

For the RBDO formulation, the probabilistic constraint in Eq. (2) can be further expressed either using an approximate statistical moment or through inverse transformations in three alternative ways^{1-5,7}:

$$G_i(\boldsymbol{\mu}) + k\sigma_{G_i} \leq 0 \quad (4)$$

$$\beta_i - \beta_{s_i} = \beta_i - \left\{ -\Phi^{-1}[F_{G_i}(0)] \right\} \leq 0 \quad (5)$$

$$G_{p_i} = F_{G_i}^{-1}[\Phi(\beta_i)] \leq 0 \quad (6)$$

where k is the target reliability constant, σ_{G_i} is the approximate statistical second moment, and β_{s_i} and G_{p_i} are the safety reliability index and the probabilistic performance measure for the i th probabilistic constraint, respectively. Because approximate first and second moments in Eq. (4) are used to describe the probabilistic constraint in the RBDO formulation, this is the AMA. Equation (5) is employed to describe the probabilistic constraint in Eq. (1) using the reliability index and is, thus, RIA. Similarly, Eq. (6) can replace the probabilistic constraint in Eq. (1) with the performance measure, which is PMA.

A. RIA in RBDO

When Eq. (5) is used, the RBDO problem is reformulated as

$$\begin{aligned} & \text{minimize} && \text{cost}(\mathbf{d}) \\ & \text{subject to} && G_{p_i}^{\text{RIA}} \equiv \beta_i - \beta_{s_i} \leq 0, \quad i = 1, \dots, np \\ & && \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \quad \mathbf{d} \in \mathbb{R}^{ndv}, \quad \mathbf{X} \in \mathbb{R}^{nr} \end{aligned} \quad (7)$$

The probabilistic constraint $G_{p_i}^{\text{RIA}}$ in RIA can be evaluated by solving the first-order reliability analysis, which is formulated as an optimization problem with an equality constraint in \mathbf{U} space:

$$\begin{aligned} & \text{minimize} && \|\mathbf{U}\| \\ & \text{subject to} && G(\mathbf{U}) = 0 \end{aligned} \quad (8)$$

In RIA, the equality constraint is the failure surface. The minimum point on the failure surface is called the most probable point (MPP) $\mathbf{u}_{G(U)=0}^*$, and the reliability index is defined by $\beta_{s,\text{FORM}} = \|\mathbf{u}_{G(U)=0}^*\|$.

For the solution of Eq. (8), either an MPP search algorithm that has been specifically developed for the first-order reliability analysis or a general optimization algorithm¹⁴ can be used. Because of its simplicity and efficiency, the Hasofer and Lind-Rackwitz and Fiessler (HL-RF) method is a popular choice for reliability analysis in RIA.

B. PMA in RBDO

According to the definition of the PMA probabilistic constraint $G_{p_i}^{\text{PMA}}$ shown in Eq. (6), the RBDO problem in Eq. (1) can be reformulated as

$$\begin{aligned} & \text{minimize} && \text{cost}(\mathbf{d}) \\ & \text{subject to} && G_{p_i}^{\text{PMA}} \equiv G_{p_i} \leq 0, \quad i = 1, \dots, np \\ & && \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \quad \mathbf{d} \in \mathbb{R}^{ndv}, \quad \mathbf{X} \in \mathbb{R}^{nr} \end{aligned} \quad (9)$$

The evaluation of the probabilistic constraint in PMA requires an inverse reliability analysis, which corresponds to the inverse problem of the reliability analysis. The first-order probabilistic constraint

$G_{p_i}^{\text{PMA}}$ can be obtained by solving the optimization problem^{1–3,7} in U space:

$$\begin{aligned} & \text{minimize} && G(U) \\ & \text{subject to} && \|U\| = \beta_t \end{aligned} \quad (10)$$

The minimum point on the target reliability surface is called MPP $\mathbf{u}_{\beta=\beta_t}^*$ with the prescribed reliability $\beta_t = \|\mathbf{u}_{\beta=\beta_t}^*\|$, and the probabilistic performance measure is defined by $G_{p,\text{FORM}} = G(\mathbf{u}_{\beta=\beta_t}^*)$. Unlike RIA, only the direction vector $\mathbf{u}_{\beta=\beta_t}^*/\|\mathbf{u}_{\beta=\beta_t}^*\|$ needs to be determined by taking advantage of the spherical equality constraint $\|U\| = \beta_t$ to find MPP $\mathbf{u}_{\beta=\beta_t}^*$.

General optimization algorithms can be employed to solve the optimization problem in Eq. (10). Although the (advanced) mean value (AMV) method is well suited for PMA due to its simplicity and efficiency, it shows instability or inefficiency for concave performance measures. Therefore, the hybrid mean value (HMV) method has been proposed to enhance numerical efficiency and stability.^{1,3,10}

C. AMA in RBDO

The RBDO problem in Eq. (1) can be redefined using AMA as

$$\begin{aligned} & \text{minimize} && \text{cost}(\mathbf{d}) \\ & \text{subject to} && G_{p_i}^{\text{AMA}} = G_i(\boldsymbol{\mu}) + k\sigma_{G_i} \leq 0, \quad i = 1, \dots, np \\ & && \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \quad \mathbf{d} \in R^{ndv}, \quad \mathbf{X} \in R^{nrv} \end{aligned} \quad (11)$$

Unlike RIA and PMA, AMA does not require a reliability analysis, that is, optimization, but requires second-order sensitivity analysis that involves a large amount of computational effort.

In the first-order second moment method, the performance function is expanded at the mean value point using the first order Taylor series as (see Refs. 4 and 5)

$$G(\mathbf{X}) \cong G(\boldsymbol{\mu}) + \sum_{i=1}^{np} \frac{\partial G(\boldsymbol{\mu})}{\partial X_i} (X_i - \mu_i) \quad (12)$$

This approximation may yield inaccurate results if \mathbf{X} is not close to the mean value $\boldsymbol{\mu}$, which occurs if the standard deviation σ_X of the random variable \mathbf{X} is large. The standard deviation of the performance function is then approximated by taking the second moment of Eq. (12) as

$$\begin{aligned} \sigma_G^2 &= \int_{-\infty}^{\infty} [G(\mathbf{X}) - G(\boldsymbol{\mu})]^2 d\mathbf{X} \\ &\cong \sum_{i=1}^{nrv} \left[\frac{\partial G(\boldsymbol{\mu})}{\partial X_i} \right]^2 \int_{-\infty}^{\infty} (X_i - \mu_i)^2 dX \\ &= \sum_{i=1}^{nrv} \left[\frac{\partial G(\boldsymbol{\mu})}{\partial X_i} \sigma_{X_i} \right]^2 \end{aligned} \quad (13)$$

In the second-order second moment method, a second-order Taylor series is used to approximate the standard deviation of the performance function (see Refs. 4 and 5).

D. Reliability Analysis Methods for RIA and PMA

The HL–RF method is a popular choice for RIA, and the HMV method, which adaptively combines the AMV and conjugate mean value (CMV) methods, is used for PMA in this paper.

1. HL–RF Method in RIA

As shown in Eq. (7), the reliability analysis in RIA is to find the minimum distance $\|U_{G_U=0}\|$ from the origin of the standard normal space to the failure surface $G(U) = 0$. The iterative algorithm of the HL–RF method is

$$\mathbf{u}^{(k+1)} = (\mathbf{u}^{(k)} \cdot \mathbf{n}^{(k)})\mathbf{n}^{(k)} + \frac{G(\mathbf{u}^{(k)})}{\|\nabla G(\mathbf{u}^{(k)})\|} \mathbf{n}^{(k)} \quad (14)$$

where $\mathbf{n}^{(k)} = [\nabla G(\mathbf{u}^{(k)})/\|\nabla G(\mathbf{u}^{(k)})\|]$ is the steepest descent direction of the performance function $G(U)$ at $\mathbf{u}^{(k)}$. The first term on

the right-hand side of Eq. (14) finds a direction with the shortest distance to the failure surface, and the second term is a correction term to reach $G(U)$.

2. HMV Method in PMA

AMV method.^{15,16} Formulation of the first-order AMV method begins with the mean value (MV) method, defined as

$$\mathbf{u}_{\text{MV}}^* = \beta_t \mathbf{n}(\mathbf{0}), \quad \mathbf{n}(\mathbf{0}) = \frac{\nabla_U G(\mathbf{0})}{\|\nabla_U G(\mathbf{0})\|} \quad (15)$$

that is, to minimize the cost $G(U)$ in Eq. (10), the normalized steepest descent direction $\mathbf{n}(\mathbf{0})$ is obtained at the MV, that is, the origin in U space. The AMV method iteratively updates the steepest descent direction at the probable point $\mathbf{u}_{\text{AMV}}^{(k)}$ starting from \mathbf{u}_{MV}^* . The iterative algorithm of the AMV method is

$$\mathbf{u}_{\text{AMV}}^{(0)} = \mathbf{0}, \quad \mathbf{u}_{\text{AMV}}^{(1)} = \mathbf{u}_{\text{AMV}}^*, \quad \mathbf{u}_{\text{AMV}}^{(k+1)} = \beta_t \mathbf{n}(\mathbf{u}_{\text{AMV}}^{(k)}) \quad (16)$$

where

$$\mathbf{n}(\mathbf{u}_{\text{AMV}}^{(k)}) = -\frac{\nabla_U G(\mathbf{u}_{\text{AMV}}^{(k)})}{\|\nabla_U G(\mathbf{u}_{\text{AMV}}^{(k)})\|} \quad (17)$$

As presented in Ref. 1, this method exhibits instability and inefficiency in solving a concave performance function because it only updates the direction using the current MPP.

a) CMV method. When applied to a concave performance function, the AMV method tends to converge very slowly or divergence due to a lack of updated information during the iterative optimization for the reliability analysis. This kind of difficulty can be overcome by using both the current and previous MPP search information as applied in the proposed CMV method.^{1,3} The new search direction is obtained by combining $\mathbf{n}(\mathbf{u}_{\text{CMV}}^{(k-2)})$, $\mathbf{n}(\mathbf{u}_{\text{CMV}}^{(k-1)})$, and $\mathbf{n}(\mathbf{u}_{\text{CMV}}^{(k)})$ with an equal weight, such that it is directed toward the diagonal of the three consecutive steepest descent directions, that is,

$$\begin{aligned} \mathbf{u}_{\text{CMV}}^{(0)} &= \mathbf{0}, \quad \mathbf{u}_{\text{CMV}}^{(1)} = \mathbf{u}_{\text{AMV}}^{(1)}, \quad \mathbf{u}_{\text{CMV}}^{(2)} = \mathbf{u}_{\text{AMV}}^{(2)} \\ \mathbf{u}_{\text{CMV}}^{(k+1)} &= \beta_t \frac{\mathbf{n}(\mathbf{u}_{\text{CMV}}^{(k)}) + \mathbf{n}(\mathbf{u}_{\text{CMV}}^{(k-1)}) + \mathbf{n}(\mathbf{u}_{\text{CMV}}^{(k-2)})}{\|\mathbf{n}(\mathbf{u}_{\text{CMV}}^{(k)}) + \mathbf{n}(\mathbf{u}_{\text{CMV}}^{(k-1)}) + \mathbf{n}(\mathbf{u}_{\text{CMV}}^{(k-2)})\|} \end{aligned} \quad \text{for } k \geq 2 \quad (18)$$

where

$$\mathbf{n}(\mathbf{u}_{\text{CMV}}^{(k)}) = \frac{\nabla_U G(\mathbf{u}_{\text{CMV}}^{(k)})}{\|\nabla_U G(\mathbf{u}_{\text{CMV}}^{(k)})\|} \quad (19)$$

Consequently, the conjugate steepest descent direction significantly improves the rate of convergence as well as the stability, as compared to the AMV method, for the concave performance function. However, the CMV method was found inefficient for the convex performance function.^{1,3} Thus, it is desirable to combine the AMV and CMV methods for the MPP search to treat both convex and concave performance functions.

b) HMV method. To select an appropriate MPP search method, the type of performance function must first be identified. The function type can be determined by employing steepest descent directions at the three consecutive iterations, as follows:

$$\zeta^{(k+1)} = (\mathbf{n}^{(k+1)} - \mathbf{n}^{(k)}) \cdot (\mathbf{n}^{(k)} - \mathbf{n}^{(k-1)})$$

$\text{sign}(\zeta^{(k+1)}) > 0$: convex type at $\mathbf{u}_{\text{HMV}}^{(k+1)}$ with respect to design \mathbf{d}

≤ 0 : concave type at $\mathbf{u}_{\text{HMV}}^{(k+1)}$ with respect to design \mathbf{d}

where $\zeta^{(k+1)}$ is the criterion for the performance function type at the $k+1$ th step and $\mathbf{n}^{(k)}$ is the steepest descent direction for a performance function at the MPP $\mathbf{u}_{\text{HMV}}^{(k)}$ at the k th iteration. Once the performance function type is defined, one of two numerical algorithms, AMV or CMV, is adaptively selected for the MPP search. This MPP search method was proposed as the HMV method in Ref. 1.

III. Studies on Probabilistic Approaches in RBDO

The numerical behavior of different probabilistic approaches in RBDO is studied in this paper. It has been shown in Refs. 1–3 that PMA is much better than RIA in terms of numerical efficiency and stability because the PMA problem is easier to solve for reliability analysis and the associated HMV method is very effective. Thus, PMA is employed in this paper, instead of RIA, to compare with AMA in the RBDO process.

Nonetheless, the RBDO methodology still has many challenges to overcome in obtaining reliability-based optimum designs for large-scale multidisciplinary problems using a limited number of design analyses. Moreover, for certain application areas, lack of design sensitivity information creates considerable difficulty in using the proposed RBDO methods. To overcome this difficulty, the RSM is often used for RBDO. Hence, the comparison studies of PMA and AMA are also carried out for two different cases: RBDO without or with employing RSM.

A. Studies on Different Probabilistic Approaches in RBDO

As described, the different constraint setup between AMA and PMA leads to a different behavior in the RBDO process, in terms of numerical efficiency, accuracy, and stability. To describe the target reliability properly, AMA must provide an appropriate value for k in Eq. (11). Here, some observations on AMA of RBDO are listed. First, it is observed that the determination of the first two moments of performance, $G(\mu)$ and σ_G , is independent of the input probabilistic distribution, as shown in Eqs. (11–13). Second, without knowing the output probabilistic distribution type, a reliability requirement is directly assigned by the first two moments of the performance function. Therefore, a nonnormal and skewed output distribution with even a small variation produces a large error when estimating the reliability. Third, another numerical error can be generated by the fact that the first two moments are obtained based on only the MV point data. Fourth, AMA involves intensive computations because

it requires the second-order sensitivity of the performance function to evaluate the sensitivity of the probabilistic constraint, whereas PMA and RIA require the first-order sensitivity. Consequently, it is difficult to accurately describe a required reliability target for RBDO, such as 6-sigma, using the AMA method.

On the other hand, PMA requires an optimization problem to be solved to estimate the reliability of the performance function. However, the reliability can be precisely estimated by advancing from the MV point to the MPP. Table 1 summarizes these discussions on numerical consideration. As shown in Table 1, PMA is much more desirable than AMA for RBDO from many numerical perspectives. Two numerical examples are presented to demonstrate these observations.

B. Numerical Example 1: Mathematical Problem

Consider the following mathematical problem for RBDO with design variables $\mathbf{d} = [d_1, d_2]^T = [\mu_1, \mu_2]^T$. The RBDO problem is defined as

$$\begin{aligned} &\text{minimize} && \text{cost}(\mathbf{d}) = d_1 + d_2 \\ &\text{subject to} && P[G_i(\mathbf{X}) \leq 0] \leq \Phi(-\beta_i), \quad i = 1, 2, 3 \\ &&& 0 \leq d_1 \leq 10, \quad 0 \leq d_2 \leq 10 \end{aligned} \quad (21)$$

where

$$G_1(\mathbf{X}) = X_1^2 X_2 / 20 - 1$$

$$G_2(\mathbf{X}) = (X_1 + X_2 - 5)^2 / 30 + (X_1 - X_2 - 12)^2 / 120 - 1$$

$$G_3(\mathbf{X}) = 80 / (X_1^2 + 8X_2 + 5) - 1 \quad (22)$$

and $\beta_i = 3.0$. The initial design is $\mathbf{d}^{(0)} = [5.0, 5.0]^T$ and random properties are $X_i \sim (\mu = 5.0, \sigma = 0.3)$, $i = 1, 2$. The sequential quadratic programming optimizer is used for this RBDO problem, and the maximum number of iterations in the reliability analysis is limited to 20. RBDO histories using PMA and AMA are presented for different target reliabilities and input distributions in Tables 2–6. To check the accuracy of evaluating the probabilistic constraints, an error by involving Monte Carlo simulation (MCS) with one-million-sample size is defined for active probabilistic constraints at the

Table 1 Comparison of AMA and PMA for RBDO

AMA	PMA
Second-order design sensitivity is required for design optimization.	First-order design sensitivity is necessary for design optimization.
Difficult to describe precisely a required target reliability (i.e., 6 sigma) using k in general RBDO.	Easy to describe a required target reliability using β_i ($= 6 \Rightarrow 6$ -sigma design).
Not capable of handling different input random variable distribution types.	Capable of dealing with all input random variable distribution types.
Inaccurate for nonnormal and skewed output distribution even with small variation.	Accurate for any type of output distribution even with large variation. ³
Larger error in reliability estimation for higher target reliability	Small error in reliability estimation for higher target reliability ($\geq 99\%$ reliability).
Not easy to integrate with RSM.	Ideally suited for integration with the RSM.
Not suitable for multidisciplinary RBDO	Very much suitable for multidisciplinary RBDO.

Table 2 Different approaches 1-sigma RBDO

Distribution	Approach	Cost	d_1	d_2	β_{MCS}^1	β_{MCS}^2	No. of analyses
Normal	PMA	5.689	3.183	2.506	0.989	1.111	30/30
	AMA	5.715	3.201	2.514	1.049	1.105	7/7/5
Lognormal	PMA	5.698	3.191	2.507	1.024	1.117	32/32
	AMA	5.749	3.215	2.533	1.149	1.149	7/7/5
Weibul	PMA	5.646	3.156	2.490	0.902	1.121	37/37
	AMA	5.646	3.165	2.481	0.910	1.083	7/7/5
Gumbel	PMA	5.689	3.202	2.486	1.105	1.116	40/40
	AMA	5.813	3.251	2.563	1.516	1.281	7/7/5
Uniform	PMA	5.821	3.210	2.611	1.194	1.260	41/41
	AMA	5.715	3.201	2.514	0.985	1.005	7/7/5

Table 3 Different approaches 2-sigma RBDO

Distribution	Approach	Cost	d_1	d_2	β_{MCS}^1	β_{MCS}^2	No. of analyses
Normal	PMA	6.202	3.297	2.905	1.993	2.074	34/34
	AMA	6.345	3.375	2.971	2.318	2.173	7/7/4
Lognormal	PMA	6.156	3.290	2.867	2.030	2.082	36/36
	AMA	6.376	3.389	2.986	2.587	2.343	7/7/5
Weibul	PMA	6.329	3.318	3.011	1.914	2.087	42/42
	AMA	6.272	3.338	2.936	1.875	1.902	7/7/4
Gumbel	PMA	6.022	3.243	2.779	2.104	2.109	47/47
	AMA	6.447	3.424	3.020	4.013	2.998	7/7/4
Uniform	PMA	6.196	3.302	2.894	2.275	2.308	80/80
	AMA	6.345	3.375	2.971	4.936	2.830	7/7/4

Table 4 Different approaches 3-sigma RBDO

Distribution	Approach	Cost	d_1	d_2	β_{MCS}^1	β_{MCS}^2	No. of analyses
Normal	PMA	6.731	3.441	3.290	2.993	3.063	49/49
	AMA	7.084	3.636	3.449	3.815	3.355	7/7/4
Lognormal	PMA	6.597	3.404	3.192	3.022	3.084	41/41
	AMA	7.110	3.647	3.463	4.494	3.866	7/7/4
Weibul	PMA	7.230	3.581	3.649	2.954	3.068	51/51
	AMA	7.008	3.597	3.411	2.812	2.624	7/7/4
Gumbel	PMA	6.298	3.285	3.013	3.095	3.127	62/62
	AMA	7.183	3.684	3.499	>6.00	>6.00	7/7/4
Uniform	PMA	6.307	3.335	2.972	3.006	3.102	114/114
	AMA	7.084	3.636	3.449	>6.00	>6.00	7/7/4

Table 5 Different approaches 4-sigma RBDO

Distribution	Approach	Cost	d_1	d_2	β_{MCS}^1	β_{MCS}^2	No. of analyses
Normal	PMA	7.270	3.610	3.660	4.009	4.081	49/49
	AMA	7.896	3.973	3.931	>6.00	4.685	10/10/5
Lognormal	PMA	7.017	3.530	3.487	4.032	4.098	49/49
	AMA	7.919	3.984	3.942	>6.00	>6.00	10/10/5
Weibul	PMA	8.328	3.949	4.379	3.957	4.086	59/59
	AMA	7.828	3.934	3.893	3.704	3.279	10/10/6
Gumbel	PMA	6.524	3.333	3.191	4.046	4.161	85/85
	AMA	7.995	4.022	3.979	>6.00	>6.00	10/10/5
Uniform	PMA	6.338	3.347	2.990	3.785	3.644	116/116
	AMA	7.896	3.973	3.931	>6.00	>6.00	10/10/5

Table 6 Different approaches 5-sigma RBDO

Distribution	Approach	Cost	d_1	d_2	β_{MCS}^1	β_{MCS}^2	No. of analyses
Normal	PMA	7.817	3.799	4.017	5.000	5.000	58/58
	AMA			Diverged			N.A.
Lognormal	PMA	7.417	3.663	3.754	5.004	5.001	56/56
	AMA			Diverged			N.A.
Weibul	PMA	9.562	4.409	5.153	5.003	5.000	69/69
	AMA			Diverged			N.A.
Gumbel	PMA	6.703	3.399	3.304	5.000	5.000	192/192
	AMA			Diverged			N.A.
Uniform	PMA	6.357	3.354	3.003	5.004	5.002	112/112
	AMA			Diverged			N.A.

optimum design as follows:

$$\left| (\beta_{MCS}^i - \beta_i) / \beta_i \right| \times 100\% \quad (23)$$

where β_{MCS}^i is the reliability of the i th performance function by MCS for the target reliability β_i .

In Tables 2–6, PMA and AMA are compared in terms of numerical accuracy, stability, efficiency, and numerical simplicity. Five different distributions are used for five different target reliabilities, from 1 to 5 sigma. In Tables 2–6, the sixth and seventh columns are the reliability evaluated by MCS at the optimum design (fourth and fifth columns) for a target reliability β_i . The last column shows numerical efficiency and simplicity. Because PMA requires only function and gradient calls, the last column in Tables 2–6 has two

numbers that correspond to the number of function and gradient calls at each iteration, respectively. To the contrary, AMA has three numbers in the last column of Tables 2–6 corresponding to the number of functions, gradient, and Hessian calls at each iteration, respectively. Thus, the AMA-based RBDO process becomes much more expensive and complicated due to the second-order design and random sensitivity analysis. First, AMA diverges when carrying out RBDO, as shown in Table 6. Because of the inaccuracy of defining probabilistic constraints in RBDO, AMA may have no feasible region for 5-sigma reliability, whereas PMA has a feasible region with accurate probabilistic constraint.

Based on Tables 2–6, Figs. 1 and 2 are shown to address the error in evaluating probabilistic constraints in RBDO. The results in Figs. 1 and 2 are the average error calculated using Eq. (23). Note

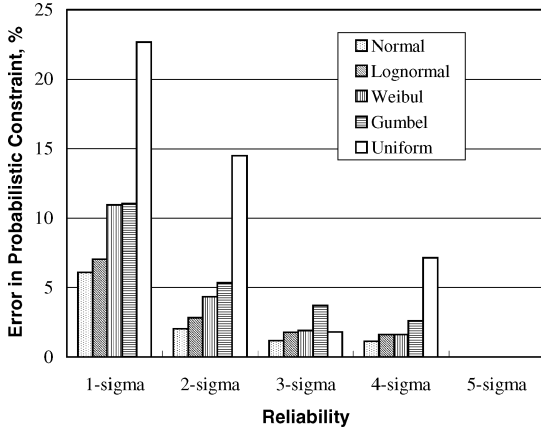


Fig. 1 Error in probabilistic constraint for PMA-based RBDO in example 1.

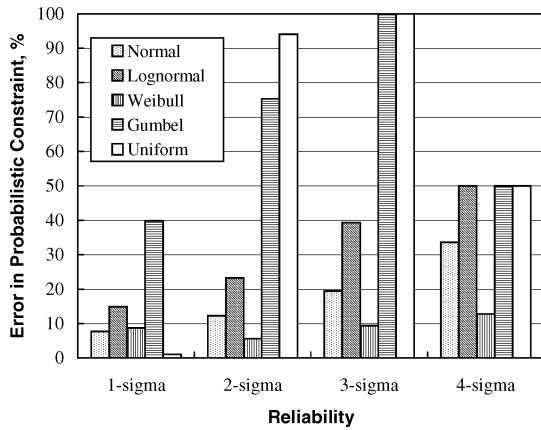


Fig. 2 Error in probabilistic constraint for AMA-based RBDO in example 1.

that, in PMA, the error is exponentially decreasing as the target reliability is getting larger because the joint probability density function to be integrated is exponentially decayed toward the tail. In contrast, larger target reliability produces larger error in AMA because the mean-based evaluation of the probabilistic constraint yields less accuracy with larger target reliability. As shown in Tables 4–6, β_{MCS}^1 and β_{MCS}^2 at some optimum designs of AMA are shown to be greater than 6 sigma. That is, one million samples of MCS at optimum designs of AMA do not contain a failure, which results in more than 6 sigma. Therefore, in Fig. 2, it is shown that Gumbel and uniform distributions for 3 sigma produce more than 100% error and three distributions (lognormal, Gumbel, and uniform) for 4 sigma produce much more than 50% error, based on the error measure in Eq. (23) such as $|(\beta_{MCS}^i - 4.0)/4.0| \times 100\% > 50\%$ with $\beta_{MCS}^i > 6.0$. Because of the restriction of sample size in MCS, the error that is measured up to 50% for 4 sigma is supposed to be much greater than 50%.

Different input distributions result in different behavior of the reliability errors in both PMA and AMA. As discussed in Ref. 3, the Gumbel and uniform distributions are very highly nonlinear compared to others, causing larger errors.

C. Numerical Example 2: Crashworthiness Problem for Side Impact

A large-scale model of a vehicle side impact is employed, as shown in Fig. 3. The side crash model includes a full-vehicle finite element (FE) structural model, an FE side impact dummy model, and an FE deformable side impact barrier model. The system model consists of 85,941 shell elements and 96,122 nodes. In the FE simulation of the side impact event, the barrier has an initial velocity of 49.89 km/h (31 mph) as it impacts the vehicle structure. The CPU time for one nonlinear FE simulation using the RADIOSS software is approximately 20 h on a SGI Origin 2000.

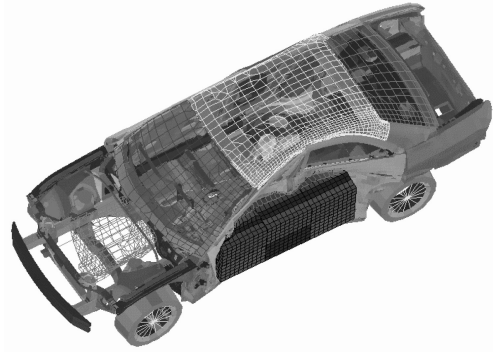


Fig. 3 Vehicle side impact model.

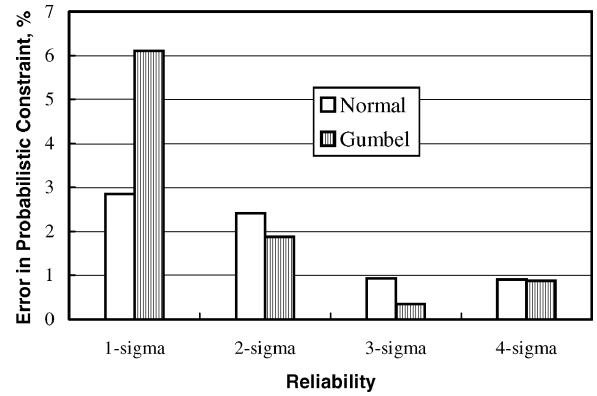


Fig. 4 Error in probabilistic constraint for PMA-based RBDO in example 1.

The design objective is to enhance side impact crash performance while minimizing vehicle weight. The RBDO problem of crashworthiness for side impact can be defined as

$$\begin{aligned}
 &\text{minimize} && \text{weight}(\mathbf{d}) \\
 &\text{subject to} && P(F_{\text{abdomen}} \leq 1.0 \text{ kN}) \leq 90\% \\
 &&& P(V C_i \leq 0.32 \text{ m/s}) \leq 90\%, \quad i = 1, 2, 3 \\
 &&& P(d_j^{\text{rib}} \leq 32 \text{ mm}) \leq 90\%, \quad j = 1, 2, 3 \\
 &&& P(F_{\text{pubic symphysis}} \leq 4.0 \text{ kN}) \leq 90\% \\
 &&& P(V_{\text{B-pillar}} \leq 10 \text{ mm/ms}) \leq 90\% \\
 &&& P(V_{\text{front door}} \leq 15.7 \text{ mm/ms}) \leq 90\% \\
 &&& \mathbf{d}^L \leq \mathbf{d} \leq \mathbf{d}^U, \quad \mathbf{d} \in \mathbb{R}^9, \quad \mathbf{X} \in \mathbb{R}^{11} \quad (24)
 \end{aligned}$$

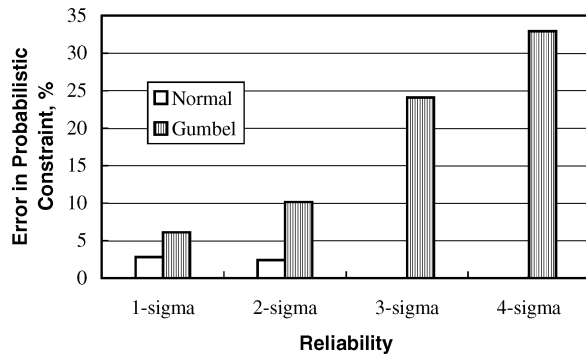
with 9 design and 11 random parameters, which are defined in Table 7.

Optimal Latin hypercube sampling with a total of 33 runs was used to generate a sample of design points for construction of the stepwise regression response surface.⁹ The explicit response used in the RBDO is summarized, and physical meanings of all responses are well described in Ref. 13. In this study, the explicit approximations of responses are regarded as exact responses of vehicle side impact to demonstrate the findings of this paper.

Errors of PMA and AMA in evaluating probabilistic constraints in RBDO are shown in Figs. 4 and 5. Again, only active probabilistic constraints in RBDO are employed to measure the error. Similar to example 1, the numerical error of PMA in Fig. 4 decreases exponentially with higher target reliability because the joint probability density function to be integrated decays exponentially as it goes toward a tail. On the other hand, a higher target reliability produces a greater error in AMA because the mean-based evaluation of probabilistic constraints becomes less accurate with higher target reliability. Note that AMA yields a very small error for a normal input

Table 7 Properties of design and random parameters of vehicle side impact model

Random variable	Standard deviation	Distribution	Design	d^L	d	d^U
1 (B-pillar inner)	0.030	Normal	1	0.500	1.000	1.500
2 (B-pillar reinforce)	0.030	Normal	2	0.450	1.000	1.350
3 (Floor side inner)	0.030	Normal	3	0.500	1.000	1.500
4 (Cross member)	0.030	Normal	4	0.500	1.000	1.500
5 (Door beam)	0.050	Normal	5	0.875	2.000	2.625
6 (Door belt line)	0.030	Normal	6	0.400	1.000	1.200
7 (Roof rail)	0.030	Normal	7	0.400	1.000	1.200
8 (Material for floor side inner)	0.006	Normal	8	0.192	0.300	0.345
9 (Material for floor side)	0.006	Normal	9	0.192	0.300	0.345
10 (Barrier height)	10.0	Normal	10th and 11th random variables not regarded as a design variable.			
11 (Barrier hitting)	10.0	Normal				

**Fig. 5 Error in probabilistic constraint for AMA-based RBDO in example 1.**

distribution because the active constraints are bilinear responses due to the specific RSM used, and thus, the output probability distribution is very close to a normal distribution. This raises an interesting question of using a certain RSM because the response surface could change the statistical characteristics of the output performance function.

In this example, two types of probabilistic distributions, normal and Gumbel, are used for four different probability levels, 1–4 sigma. As explained earlier, it can be seen that nonnormal and skewed output distributions, such as Gumbel, with even small variations yields large errors in the evaluation of reliability.

D. Studies on RBDO Integrated with RSM

With increasing requirements for solving large-scale problems, it is desirable to integrate RBDO with a RSM.^{10,17} Moreover, for certain application areas, a lack of design sensitivity information creates considerable difficulty in using the proposed RBDO methods. It has been shown in the literature that PMA is much more effective than RIA when RBDO is integrated with RSM.¹⁰ In this section, the comparative study between PMA and AMA is extended to RSM-based RBDO.

In addition to numerical considerations observed in the preceding section, more comparisons can be made for PMA and AMA in RSM-based RBDO. For RSM-based design optimization, there are two approaches: local and global response surface representations. For RBDO, it is strongly recommended to use a local RSM, due to the significant effect that an inaccurate response surface can have on reliability analysis, such that the response surface could change the statistical characteristics of the output performance function. RSM-based RBDO can share the same local window as design and analysis. In this case, inaccuracies in evaluating probabilistic constraints in AMA cause difficulties in determining the proper size of the analysis window. In addition, AMA requires second-order sensitivities for design optimization, which will be challenging to obtain because the RSM that is based on function evaluation will not provide accurate second-order sensitivities. A clear distinction is made, based on the preceding discussion, which is summarized in Table 8.

Table 8 Comparison of AMA and PMA for RSM-based RBDO

AMA	PMA
Difficult to define the size of analysis windows, which may have to be of different size than the design window	Same DOE window can serve as design and analysis window
At least first-order sensitivity information is necessary to generate an accurate response surface for RSM-based RBDO	Possible to perform RSM-based RBDO with only function evaluations

IV. Conclusions

This paper presents basic studies carried out on probabilistic approaches in RBDO, regarding various numerical considerations, providing a confident guideline in selecting the most desirable method for RBDO. It is found that PMA is accurate enough and stable at an allowable efficiency, whereas AMA has some difficulties in RBDO process such as a second-order design sensitivity required for design optimization, an inaccuracy to measure a probability of failure, and numerical instability due to its inaccuracy. Difficulties in evaluating probabilistic constraints accurately prevent AMA from performing the RBDO process effectively, compared to PMA. Furthermore, if RSM is used for RBDO, PMA is preferable to AMA because the size of the design and analysis window is well defined and second-order design sensitivities are not required for the RBDO process. Even though the findings of this paper are demonstrated through a limited investigation of analytic example and a large-scale engineering application, it provides a rigorous guideline to choose a proper method for RBDO. Consequently, PMA is quite attractive when compared to other probabilistic approaches in RBDO.

Acknowledgments

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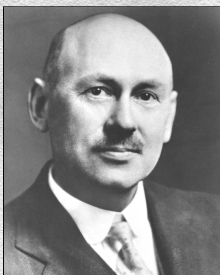
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